Session 1 - Solutions to Exercises

Exercise 1

Exercise: Prove that classical semantic consequence $\vdash$ is reflexive, transitive, and monotonic.

Solution: Let $\Gamma, \Delta \subseteq \mathcal{L}$ be arbitrary sets of sentences, and let w represent an arbitrary possible world.

- $\vdash$ **is Reflexive:** Let γ be an arbitrary sentence in Γ. Consider any world w such that $w \models \Gamma$ (meaning w satisfies every sentence in Γ). Since $\gamma \in \Gamma$, it trivially follows that $w \models \gamma$. Because w was an arbitrary world satisfying the premises, we conclude $\Gamma \vdash \gamma$.
- $\vdash$ **is Transitive:** Assume $\Gamma \vdash \delta$ for all $\delta \in \Delta$, and assume $\Delta \vdash \phi$. Consider an arbitrary world w such that $w \models \Gamma$. By our first assumption, $w \models \delta$ for every $\delta \in \Delta$. This is logically equivalent to saying $w \models \Delta$. Because $w \models \Delta$, our second assumption ($\Delta \vdash \phi$) guarantees that $w \models \phi$. Since w was an arbitrary world satisfying Γ, we conclude $\Gamma \vdash \phi$.
- $\vdash$ **is Monotonic:** Assume $\Gamma \subseteq \Delta$ and $\Gamma \vdash \phi$. Consider an arbitrary world w such that $w \models \Delta$. Because $\Gamma \subseteq \Delta$, any world that satisfies all of Δ must also satisfy all of Γ; therefore, $w \models \Gamma$. Since $\Gamma \vdash \phi$, it follows that $w \models \phi$. Since w was an arbitrary world satisfying Δ, we conclude $\Delta \vdash \phi$.

Exercise 2

Exercise: Prove that the classical consequence operator Cn satisfies reflexivity, transitivity, monotonicity, and idempotence.

Solution: Reflexivity, transitivity, and monotonicity for Cn follow directly from the corresponding properties of the semantic consequence relation $\vdash$ proved in Exercise 1, simply by translating them through the definition $Cn(\Gamma) := \{\phi \in \mathcal{L} : \Gamma \vdash \phi\}$. We will explicitly prove the fourth property: Idempotence.

To prove set equality, we must prove mutual inclusion.

($\subseteq$) **Prove $Cn(\Gamma) \subseteq Cn(Cn(\Gamma))$:** This direction follows immediately from the reflexivity of Cn. Reflexivity states that for any set X, $X \subseteq Cn(X)$. By substituting $Cn(\Gamma)$ for X, we immediately get $Cn(\Gamma) \subseteq Cn(Cn(\Gamma))$.

($\supseteq$) **Prove $Cn(Cn(\Gamma)) \subseteq Cn(\Gamma)$:** Let ϕ be an arbitrary sentence such that $\phi \in Cn(Cn(\Gamma))$. By the definition of the consequence operator, this means $Cn(\Gamma) \vdash \phi$. Furthermore, by definition, for every sentence $\psi \in Cn(\Gamma)$, it is true that $\Gamma \vdash \psi$. We can now apply the transitivity of $\vdash$: since $\Gamma \vdash \psi$ for all premises $\psi \in Cn(\Gamma)$, and $Cn(\Gamma) \vdash \phi$, it follows that $\Gamma \vdash \phi$. Therefore, $\phi \in Cn(\Gamma)$. Since ϕ was arbitrary, $Cn(Cn(\Gamma)) \subseteq Cn(\Gamma)$.

Mutual inclusion establishes the identity $Cn(\Gamma) = Cn(Cn(\Gamma))$. □

Exercise 3

Exercise: The properties of consequence relations $\rhd$ and consequence operators C are not all logically independent. Explore their logical interactions by proving the following claims:

1. **Reflexivity + Transitivity $\implies$ Monotonicity:** Prove that if a consequence relation $\rhd$ (or operator C) is reflexive and transitive, it must necessarily be monotonic.
2. **Monotonicity + Idempotence $\implies$ Transitivity:** For a consequence operator C, prove that if C is monotonic and idempotent, then C is transitive.
3. **Reflexivity + Transitivity $\implies$ Idempotence:** For a consequence operator C, prove that if C is reflexive and transitive, then C is idempotent.
4. **Reflexivity + Monotonicity $\not\implies$ Transitivity:** Provide a counterexample (e.g., a restricted consequence operator on $\mathcal{L}$) that is reflexive and monotonic, but fails to be transitive.

(*N.b.*, 4 implies that **Reflexivity + Monotonicity $\not\implies$ Idempotence**.)

Solution:

1. Reflexivity + Transitivity $\implies$ Monotonicity. Suppose $\rhd$ is reflexive and transitive, and suppose $\Gamma \subseteq \Delta$ and $\Gamma \rhd \phi$. By reflexivity, $\Delta \rhd \delta$ for all $\delta \in \Delta$. Thus, in particular, $\Delta \rhd \gamma$ for all $\gamma \in \Gamma$, since $\Gamma \subseteq \Delta$. Since $\Gamma \rhd \phi$, the transitivity of $\rhd$ yields $\Delta \rhd \phi$. Thus, $\rhd$ is monotonic.

The proof is analogous for any consequence operator C. Suppose $\Gamma \subseteq \Delta$. By reflexivity, $\Delta \subseteq C(\Delta)$, and hence $\Gamma \subseteq C(\Delta)$. By the transitivity of C, since $\Gamma \subseteq C(\Delta)$, it follows that $C(\Gamma) \subseteq C(\Delta)$. Therefore, C is monotonic.

2. Monotonicity + Idempotence $\implies$ Transitivity. Suppose C is monotonic and idempotent. Suppose also that $\Gamma \subseteq C(\Delta)$. By the monotonicity of C, $C(\Gamma) \subseteq C(C(\Delta))$. By idempotence, $C(C(\Delta)) = C(\Delta)$, which yields $C(\Gamma) \subseteq C(\Delta)$. Therefore, C is transitive.

3. Reflexivity + Transitivity $\implies$ Idempotence. Suppose C is reflexive and transitive. By reflexivity, $C(\Gamma) \subseteq C(C(\Gamma))$. We now need to prove that $C(C(\Gamma)) \subseteq C(\Gamma)$. By reflexivity, $C(\Gamma) \subseteq C(\Gamma)$ holds trivially. By transitivity, if $C(\Gamma) \subseteq C(\Gamma)$, then $C(C(\Gamma)) \subseteq C(\Gamma)$. Therefore, $C(\Gamma) = C(C(\Gamma))$.

Reflexivity + Monotonicity $\implies$ (Transitivity $\iff$ Idempotence)

As a result of (2) and (3), it follows immediately that, given C is reflexive and monotonic, C is idempotent iff C is transtive

4. Reflexivity + Monotonicity $\not\Longrightarrow$ Transitivity. Let $\mathcal{L}$ be a standard propositional language. Define a consequence operator C that takes a set of sentences Γ and adds the negation of each sentence in Γ:

$$C(\Gamma) = \Gamma \cup \{\neg\phi : \phi \in \Gamma\}$$

Reflexivity holds trivially since $\Gamma \subseteq C(\Gamma)$. Monotonicity holds because if $\Gamma \subseteq \Delta$, then $\{\neg\phi : \phi \in \Gamma\} \subseteq \{\neg\phi : \phi \in \Delta\}$, which ensures $C(\Gamma) \subseteq C(\Delta)$.

However, C fails Transitivity. Let $\Gamma = \{p\}$. Then $C(\Gamma) = \{p, \neg p\}$. If we apply the operator a second time, we obtain $C(C(\Gamma)) = \{p, \neg p, \neg\neg p\}$. Since $\neg\neg p \notin C(\Gamma)$, we have $C(C(\Gamma)) \nsubseteq C(\Gamma)$. Thus, C is not idempotent. Given that Reflexivity + Transitivity $\Longrightarrow$ Idempotence (as proven in step 3), the failure of idempotence logically guarantees the failure of transitivity. To verify this directly against the definition of transitivity: let $X = \{p, \neg p\}$ and $Y = \{p\}$. We have $X \subseteq C(Y)$, but $C(X) \nsubseteq C(Y)$.

Reflexivity + Monotonicity $\not\Longrightarrow$ Idempotence

This is directly shown in the example above, but it also follows from (3).

Exercise 4

Exercise:

1. **Reflexivity + Cut + Idempotence $\not\Longrightarrow$ Monotonicity:** Provide a counterexample demonstrating that a consequence relation $\triangleright$ (or operator C) can be reflexive, cumulatively transitive, and idempotent, yet fail to be monotonic.
2. **Reflexivity + Monotonicity $\Longrightarrow$ (Transitivity $\Longleftrightarrow$ Cut):** Prove this logical equivalence.
3. **Reflexivity + Monotonicity $\Longrightarrow$ (Idempotence $\Longleftrightarrow$ Cut):** Prove this logical equivalence.

Solution:

1. Reflexivity + Cut + Idempotence $\not\Longrightarrow$ Monotonicity: Let $\mathcal{L}$ be a standard propositional language. Define a consequence operator C such that:

- If Γ is consistent, $C(\Gamma) = Cn(\Gamma)$ (where Cn is classical consequence).
- If Γ is inconsistent, $C(\Gamma) = \Gamma$.

Reflexivity: C is reflexive because Cn is reflexive (so $\Gamma \subseteq Cn(\Gamma)$) and trivially $\Gamma \subseteq \Gamma$.

Idempotence: If Γ is consistent, $Cn(\Gamma)$ is consistent, so $C(C(\Gamma)) = C(Cn(\Gamma)) = Cn(Cn(\Gamma)) = Cn(\Gamma) = C(\Gamma)$. If Γ is inconsistent, $C(C(\Gamma)) = C(\Gamma) = \Gamma$. Thus, C is idempotent.

Cut: Suppose $\Gamma \subseteq \Delta \subseteq C(\Gamma)$. If Γ is consistent, then $C(\Gamma) = Cn(\Gamma)$. Since $\Delta \subseteq Cn(\Gamma)$, Δ must also be consistent. Therefore, $C(\Delta) = Cn(\Delta)$. Since Cn satisfies Cut, $Cn(\Delta) \subseteq Cn(\Gamma)$, meaning $C(\Delta) \subseteq C(\Gamma)$. If Γ is inconsistent, $C(\Gamma) = \Gamma$. The assumption $\Gamma \subseteq \Delta \subseteq \Gamma$ implies $\Gamma = \Delta$. Thus $C(\Delta) = C(\Gamma)$, which satisfies $C(\Delta) \subseteq C(\Gamma)$.

Failure of Monotonicity: Let $\Gamma = \{p\}$ and $\Delta = \{p, \neg p\}$. Γ is consistent, so $C(\Gamma) = Cn(\{p\})$, which contains $p \vee q$. Δ is inconsistent, so $C(\Delta) = \Delta = \{p, \neg p\}$. We have $\Gamma \subseteq \Delta$, but $p \vee q \in C(\Gamma)$ and $p \vee q \notin C(\Delta)$. Thus, $C(\Gamma) \nsubseteq C(\Delta)$, meaning C is not monotonic.

2. Reflexivity + Monotonicity $\implies$ (Transitivity $\iff$ Cut). Suppose that C satisfies reflexivity and monotonicity.

(2.a) Suppose that C satisfies Cut, and also that $\Delta \subseteq C(\Gamma)$. We need to prove that $C(\Delta) \subseteq C(\Gamma)$. By reflexivity, $\Gamma \subseteq C(\Gamma)$. Since we assumed $\Delta \subseteq C(\Gamma)$, it follows that their union is also a subset of $C(\Gamma)$, yielding $\Gamma \subseteq \Gamma \cup \Delta \subseteq C(\Gamma)$. By Cut, this implies $C(\Gamma \cup \Delta) \subseteq C(\Gamma)$. Since $\Delta \subseteq \Gamma \cup \Delta$, monotonicity guarantees that $C(\Delta) \subseteq C(\Gamma \cup \Delta)$. By transitivity of the subset relation, $C(\Delta) \subseteq C(\Gamma)$. Thus, C is transitive.

(2.b) Suppose that C satisfies transitivity, and also that $\Gamma \subseteq \Delta \subseteq C(\Gamma)$. We need to prove that $C(\Delta) \subseteq C(\Gamma)$. From our assumption, we have $\Delta \subseteq C(\Gamma)$. By the definition of transitivity, if $\Delta \subseteq C(\Gamma)$, then $C(\Delta) \subseteq C(\Gamma)$. Thus, C satisfies Cut.

3. Reflexivity + Monotonicity $\implies$ (Idempotence $\iff$ Cut). Suppose that C satisfies reflexivity and monotonicity.

(3.a) Suppose C is idempotent, and let $\Gamma \subseteq \Delta \subseteq C(\Gamma)$. We must prove $C(\Delta) \subseteq C(\Gamma)$. From the assumption, $\Delta \subseteq C(\Gamma)$. By monotonicity, $C(\Delta) \subseteq C(C(\Gamma))$. By idempotence, $C(C(\Gamma)) = C(\Gamma)$. Substituting this equality yields $C(\Delta) \subseteq C(\Gamma)$. Thus, C satisfies Cut.

(3.b) Suppose C satisfies Cut. We must prove $C(C(\Gamma)) = C(\Gamma)$. By reflexivity, $C(\Gamma) \subseteq C(C(\Gamma))$. It remains to prove that $C(C(\Gamma)) \subseteq C(\Gamma)$. By reflexivity, we know $\Gamma \subseteq C(\Gamma)$. Let $\Delta = C(\Gamma)$. We therefore have $\Gamma \subseteq \Delta \subseteq C(\Gamma)$. Applying Cut yields $C(\Delta) \subseteq C(\Gamma)$. Substituting $C(\Gamma)$ back for Δ, we obtain $C(C(\Gamma)) \subseteq C(\Gamma)$. Having established mutual inclusion, $C(C(\Gamma)) = C(\Gamma)$. Thus, C is idempotent.

Exercise 5